MATHEMATICS

AARYAN GURUKUL
HIGHER SECONDARY SCHOOL

STUDY MATERIAL
FOR FOUNDATION COURSE – 2020

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SET THEORY

- Concept of set was given by German mathematician George cantor.
- A set is defined as collection of well-defined objects.
- A set is represented by a capital letter and the element of a set is denoted by small letter.
- If x is an element of a set S then we write it as $x \in S$ (read as x belongs to S) on the other hand if x is not an element of S we write it as $x \notin S$.
- A set can be expressed in two methods, they are
  (i) Tabular or Roster method
  (ii) Set-builder method.
- In Tabular method we have to write all the elements of a set in a curly bracket, like
  $S = \{1, 2, 3, 4, 5\}$ $A = \{a, e, i, o, u\}$, $N = \{1, 2, 3, 4, \ldots.\}$.
- In set-builder method we have to write a set by observing general characteristics of its element. In the above example
  $S = \{x \mid x \text{ is a natural number and } 1 \le x \le 5\}$
  $A = \{x \mid x \text{ is vowel of English alphabets}\}$
  $N = \{x \mid x \text{ is a natural number}\}$
- **Types of set :**
  (i) Finite Set - A set having countable number of elements. If A is a finite set then its number of element is denoted by $|A|$.
  (ii) Infinite Set – A set whose number of element can not be counted.
  (iii) Empty or Null set – A set having no element. It is denoted by ϕ (p h i) or { }.
  (iv) Subset – If A and B are two sets and all the elements of A are also elements of B then A is subset of B and we write it as $A \subset B$ or $B \supset A$. In other words we say B contains A .
  Mathematically $A \subset B$ means $x \in A \Rightarrow x \in B$.
  Remember - $\phi \subset A$, $A \subset A$.
  (v) Equal Set – For two sets A and B if $A \subset B$ as well as $B \subset A$ then A and B are equal we write it as $A = B$.
  (vi) Universal Set - If a number of set are subset of a fixed set (E) then E is called universal set.
  Let E = Set of students of a school.
  A = Set of boys of a school.
  B = Set of girls of a school.
  Hence $A \subset E$, $B \subset E$ so E is our universal set.

N.B.: - In venn diagram universal set is represented by a rectangle and subsets of the universal set are represented by circle or any closed figure inside the rectangle.

- Set Operations – There are three basic operations among set they are
  (i) Union (ii) Intersection (iii) Difference

(i) UNION – If A and B are two non-empty sets then their union is denoted by $A \cup B$. Mathematically $A \cup B$ is the set consisting of all the elements of A as well as B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Properties :

- If $A \subset B$ then $A \cup B = B$.
  If $B \subset A$ then $A \cup B = A$.
  For any set A, $A \cup A = A$, $A \cup \phi = A$, $A \subset A \cup B$, $B \subset A \cup B$.
- Commutative Law : $A \cup B = B \cup A$
  Associative Law : $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) INTERSECTION : For A and B their intersection is $A \cap B$. Mathematically $A \cap B$ is the set consisting of the elements which belongs to both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

If $A \cap B \neq \phi$ then A and B are joint sets.

If $A \cap B = \phi$ then A and B are disjoint sets.

Properties :

- If $A \subset B$ then $A \cap B = A$, If $B \subset A$ then $A \cap B = B$.
  For any set A, $A \cap A = A$, $A \cap \phi = \phi$, $A \cap B \subset A$, $A \cap B \subset B$.
- Commutative Law : $A \cap B = B \cap A$
  Associative Law : $A \cap (B \cap C) = (A \cap B) \cap C$.

DISTRIBUTIVE LAW:

(1) Union is distributive over intersection i.e., $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(2) Intersection is distributive over union i.e., $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

- **DIFFERENCE OF TWO SETS:**
  Let A and B are two sets, then $A - B$ is the set of these elements which belongs to A but not B.
  $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
  $B - A = \{x \mid x \in B \text{ and } x \notin A\}$
- $A - A = \phi$, $A - \phi = A$, $A - B \subset A$, $B - A \subset B$,
  $(A - B) \cap (B - A) = \phi$, $(A - B) \cap (A \cap B) = \phi$, $(B - A) \cap (A \cap B) = \phi$.
  So $(A - B)$, $(A \cap B)$, $(B - A)$ are pairwise disjoint sets.
- $A - B \neq B - A$, $A - (B - C) \neq (A - B) - C$

Symmetric – Difference: - Let A and B are two sets then union of (A – B) and (B – A) is called symmetric difference of A and B. It is denoted by $A \Delta B$.

$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

→ $A \Delta B = B \Delta A$, $A \Delta (B \Delta C) \neq (A \Delta B) \Delta C$.

→ Complement of a set : - Let E is an universal set and A be any subset of E. Then the set of elements belonging to E but not A is called complement of A denoted by A′.

$A' = \{ x \mid x \in E \text{ and } x \notin A\}$.

→ $A \cap A' = \phi$, $A \cup A' = E$, $(A')' = A$, $\phi' = E$, $E' = \phi$.

Demorgan's Laws :

Let A, B are any two subsets of E. Then $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$.

→ For any two sets A and B

(i) $|A \cup B| = |A| + |B| - |A \cap B|$, if A & B are joint sets.

$= |A| + |B|$, if A and B are disjoint.

(ii) $|A \Delta B| = |A| + |B| - 2\,|A \cap B|$

$= |A \cup B| - |A \cap B|$

Cartesian Product of two sets : -

For any two sets A and B, $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

→ If $|A| = m$, $|B| = n$ then $|A \times B| = |B \times A| = mn$.

→ $A \times \phi = \phi \times A = \phi \times \phi = \phi$.

Power set of a set A : -

If A is a finite set. Then power set of A denoted by P (A) is the set of all possible subsets of A.

→ If $|A| = n$ then $|P(A)| = 2^n$.

Venn Diagram :

A set is expressed through closed figure like circle, rectangle etc.

Universal set 'E' is represented by a rectangle and any subset of E is represented by a circle.

Set of numbers:

N = {1, 2, 3, 4, …..} = set of natural numbers.

$N^* = W = \{0, 1, 2, 3, 4, \ldots\}$ = set of whole numbers.

$Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ = set of integers.

Q = Set of rationals = $\{\frac{p}{q} \mid p, q \in Z, q \neq 0\}$

IR = Q′ = Set of irrationals = numbers which cannot be written in the form of p/q.

R = Set of real numbers = $Q \cup Q'$, $Q \cap Q' = \phi$.

Set of Even numbers = E = $\{2x \mid x \in Z\}$

Set of odd numbers = O = $\{2x + 1 \mid x \in Z\}$.

Note:

Q = Repeating nonterminating decimals.

IR = Q′ = Non repeating nonterminating decimals.

Prime numbers :

A number greater than 1 which is divisible by 1 and itself.

= {2, 3, 5, 7, 11, 13, 17, 19, ….}

1 is not a prime number.

Number of prime numbers between 1 and 1000 is 168.

Number of prime numbers between 1000 and 2000 is 135.

Number of prime numbers between 2000 and 3000 is 127.

Number of prime numbers between 3000 and 4000 is 120.

Number of prime numbers between 4000 and 5000 is 119.

So set of prime numbers is an infinite set.

Qns. Find the number of prime numbers between 1 & 100.

Note: 1 is neither prime nor composite.

2 is the only prime number which is even.

→ A number which has more than 2 divisors is called composite number eg: - 4, 6, 8, 9, …etc.

→ Expressing a composite number as product of its different prime factors is called canonical form.

Eg.: $4 = 2 \times 2 = 2^2$, $16 = 2^4$, $15 = 3 \times 5$, etc.

Important algebraic formulae:

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $a^2 - b^2 = (a + b)(a - b)$
4. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b)$
5. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b)$
6. $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)^3 - 3ab(a + b)$
7. $a^3 - b^3 = (a - b)(a^2 + ab + b^2) = (a - b)^3 + 3ab(a - b)$
8. $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$
9. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$
10. $a^3 + b^3 + c^3 - 3abc$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$$

Quadratic equation in one variable:

It is given by $ax^2 + bx + c = 0$, $a, b, c \in R$ and $a \neq 0$.

Roots of $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

which gives two roots and are denoted by α and β.

$b^2 - 4ac$ is called discriminant denoted by D. So $D = b^2 - 4ac$.

$\alpha + \beta = -\frac{b}{a}$ $\alpha\beta = \frac{c}{a}$ $\alpha - \beta = \frac{\sqrt{D}}{a}$

Formation of quadratic equation whose roots are given:

If α, β are the roots of a quadratic equation then the quadratic equation is $(x - \alpha)(x - \beta) = 0$ (or) $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

(or) x^2 – (sum of roots) x + (product of roots) = 0.

Nature of roots of a quadratic equation:

Let the quadratic equation be $ax^2 + bx + c = 0$ whose $D = b^2 - 4ac$.

(i) If $D > 0$ then roots (α and β) are real and different and if D is a perfect square then roots are rational and different.

if D is not a perfect square then roots are irrational and conjugate of each other.

e.g: If one of the root of a quadratic equation is $p + \sqrt{q}$ then the other root is $p - \sqrt{q}$.

(ii) If $D = 0$ then the roots are real and equal.

(iii) If $D < 0$ then the roots are complex (not real) and conjugate of each other.

Simultaneous Linear Equations in two unknown (variables):-

Here two equations are,

$a_1x + b_1y + c_1 = 0$ …. (1)

$a_2x + b_2y + c_2 = 0$ ….(2)

To solve the above equations we have following methods.

1. *Method of substitution : -*

(i) From equation (1) find y in terms of x and put it in equation (2) which gives a linear equation in x and on solving we get the value of x.

(ii) Put the value of x as obtained in step (1) in equation (1) or (2) to get the value of y.

(iii) Finally the solution is (x, y).

2. *Method of Elimination:*

(i) By proper manipulation eliminate y from both the equations and get the value of x.

(ii) Put the value of x as obtained in step (1) in equation (1) or (2) to get the value of y.

3. *Cross-multiplication method:*

Using the formula,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$ we can get the solution as

$$\left(x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

Condition for solvability:

(i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ unique solution, equations are consistent and independent.

(ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ infinite solution, equations are consistent and dependent.

(iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ no solution, equations are inconsistent.

INDICES

$a^n = a \times a \times a \times \ldots.. \times a$ (n times). Where $a \in R$, $n \in N$.

Here a = base n = index or exponent.

→ $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$, $(ab)^n = a^n . b^n$ $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $o^n = 0$, $1^n = 1$, $a^{-n} = \frac{1}{a^n}$, $a^0 = 1$.

→ o° = meaningless = undefined → $a^m = a^n \Rightarrow m = n$.

→ $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

LOGARITHM

If $a^x = y \Leftrightarrow \log_a^y = x$ where $a > 0$, $a \neq 1$, $y > 0$.

Ex. $\log_2^8 = 3$, $\log_2^{16}$ 4, $\log_3^3 = 1$, $\log_{10}^{1000} = 3$, $\log_5^{\sqrt{5}} = \frac{1}{2}$ etc.

$\text{Log}_a^x = 1$, $\log_a^1 = 0$.

Rules of logarithm:

1. $\log_a^{mn} = \log_a^m + \log_a^n$
2. $\log_a^{\frac{m}{n}} = \log_a^m - \log_a^n$
3. $\log_a^{m^n} = n \log_a^m$
4. $\log_a (mnp\ldots) = \log_a^m + \log_a^n + \log_a^p + \ldots.$
5. $\log_a \frac{mn}{pq} = \log_a^m + \log_a^n - \log_a^p - \log_a^q$
6. $\frac{\log a}{\log b} = \log_b^a$.

Change of base:

$\log_a^m = \log_b^m \times \log_a^b$ $\log_b^a \times \log_a^b = 1$

$\log_a^m = \log_b^m \times \log_c^b \times \log_a^c$

logarithm

natural (base e) — $\log_e^x = \ln x$

common (base 10) — $\log_{10}^x = \log x$.

log2 = 0.3010 log 3 = 0 . 4771, log 7 = 0.9451, log10=1

log 5 = 0.6990.

$\log_a^x$ is – ve if $0 < x < 1$.

$0 < \log_e^x < 1$ if $1 < x < e$.

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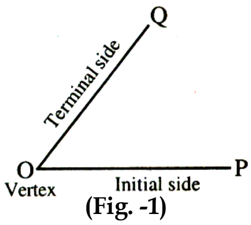
TRIGONOMETRY

Introduction:

The word trigonometry is derived from the Greek words 'trigonon' means a triangle and 'metron' means measure and hence it litereally means 'measurement of triangles". Initially, trigonometry dealt with the relationships between the sides and angles of a triangle. Now-a-days, the scope of the subject has widened and we discuss some of them.

Angle – An angle is considered as the figure obtained by rotating a given ray about its end-point. The original ray is called the initial side and the ray into which the initial side rotates is called the terminal side of the angle.



(Fig. -1)

The measure of an angle is the amount of rotation required to get the terminal side from the initial side.

If the rotation is in the clockwise sense, the angle measured is negative and it is positive if the rotation is in the anti-clockwise sense.

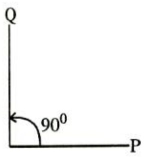
We place the vertex of the angle at the centre of a circle of some fixed radius. Then we divide the circumference of the circle into 360 equal parts, called degrees.

The number of degrees on the circumference between the initial and terminal sides of the angle is its degree measure. Hence a complete rotation describes 360°.

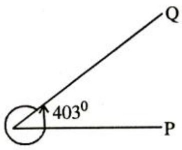
Right angle = 90° (90 degrees)

$1° = 60'$ (60 minutes), $1' = 60''$ (60 seconds)

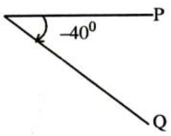
There is another unit of angular measurement called the radian measure or circular measure.



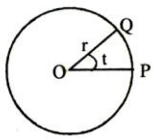
(Fig. -2)



(Fig. -3)



(Fig. -4)



(Fig. - 5)

This is based upon the fact that the ratio of the circumference of the circle to its diameter is constant.

This constant π is an irrational number which has non-recurring decimal expansion i.e., $\pi = 3.14159\ldots$, $\frac{22}{7}$ is taken as an approximate value of π. In the fig-5, if the length of the arc subtending the angle at the centre is 's' then radian measure 't' of the angle is defined to be $\frac{s}{r}$. This angle does not depend upon the radius of the circle considered.

The radian measure t of the angle formed by one complete revolution of the initial side is $\frac{2\pi r}{\pi} = 2\pi$.

Thus 2π radians = 360° or π radians = 180°.

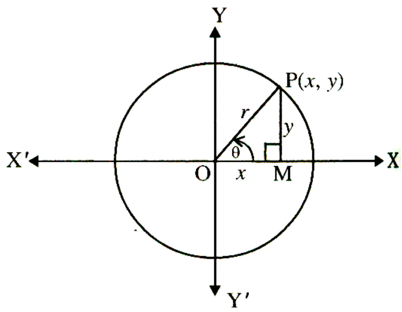
$\therefore$ 1 radian = $\frac{180°}{\pi} = 57°\ 17'\ 45''$ approximately

and 1 degree = $\frac{\pi}{180}$ radian = 0.01746 radian approximately.

Note : π does not stand for 180°. π is a real number where as π^c stands for 180°.

Trigonometrical ratios:

Let X′OX and Y′OY be the co-ordinate axes and let a revolving line starts from OX in the anti-clockwise direction and trace out an angle $\angle XOP = \theta$.



(Fig. - 6)

From P, draw PM ⊥ OX.

Let in the right angled triangle POM.

Base = OM = X,

Perpendicular distance = PM = y and hypotenuse = OP = r.

There are six possible rations among three sides of a triangle.

These six ratios are called trigonometrical ratios and defined as follows:

$\sin\theta = \frac{PM}{OP} = \frac{y}{r}$ $\quad \cos\theta = \frac{OM}{OP} = \frac{x}{r}$

$\tan\theta = \frac{PM}{OM} = \frac{y}{x}$ $\quad \cot\theta = \frac{OM}{PM} = \frac{x}{y}$

$\text{cosec}\,\theta = \frac{OP}{PM} = \frac{r}{y}$ $\quad \sec\theta = \frac{OP}{PM} = \frac{r}{x}$

Hence, the trigonometrical ratios of an angle are the numerical quantise. Each one of them represents the ratio of the length of one se to another side of a right angled triangle.

Trigonometrical identities:

We have the following three identities among the trigonometrical ratios:

(a) $\sin^2\theta + \cos^2\theta = 1$

(b) $1 + \tan^2\theta = \sec^2\theta$

(c) $1 + \cot^2\theta = \text{cosec}^2\theta$.

These are called **"Pythagorean Identities"**.

Note: Since $\sin^2\theta + \cos^2\theta = 1$,

$\therefore\ |\sin\theta| \le 1$ and $|\cos\theta| \le 1$

$\Rightarrow -1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$

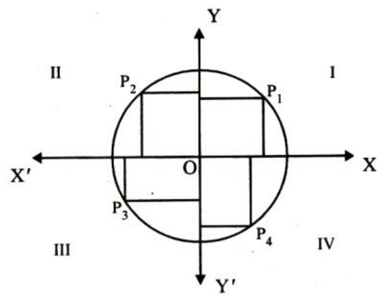
Also, $0 \le \sin^2\theta \le 1$, $\quad 0 \le \cos^2\theta \le 1$

Since, cosec $\theta = \frac{1}{\sin\theta}$, therefore cosec $\theta \ge 1$ or cosec $\theta \le -1$

Since, sec $\theta = \frac{1}{\cos\theta}$, therefore sec $\theta \ge 1$ or sec $\theta \le -1$.

Position of a Point:

Let P_1, P_2, P_3, P_4 be the position of P (x, y) in the first, 2nd, 3rd and 4th quadrants respectively. When p (x, y) lies



(Fig. -7)

(i) in the I quadrant, x > 0 and y > 0. Hence six ratios are positive.

(ii) in the II quadrant, x < 0 and y > 0. Hence sin θ and cosec θ are positive and the rest are negative.

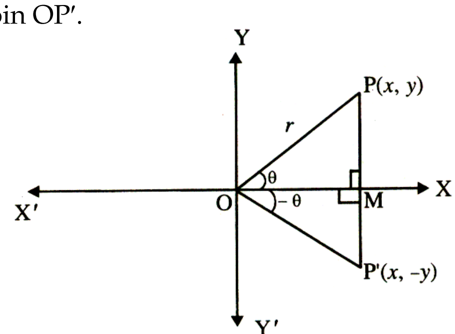
(iii) in the III quadrant, x < 0 and y < 0. Hence the ratios tan θ and cot θ are positive and the rest are negative.

(iv) in the IV quadrant, x > 0 and y < 0.

Hence cos θ and sec θ are positive and rest and negative.

Trigonometrical ratios of (– θ)

Let ∠POM=θ, in ΔPOM and OP = r, OM = x, PM = y. Now produce PM to P′ such that PM = MP′
Join OP′.



(Fig. - 8)

Hence ΔOMP ≡ ΔOMP′

With due regard to proper sign, we have

∠ P′OM = – θ,

OM = x, OP′ = r and P′ M = – y

In ΔPOM,

$$\sin(-\theta) = \frac{P'M}{OP'} = \frac{-y}{r} = -\frac{PM}{OP} = -\sin\theta.$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{x}{r} = \frac{OM}{OP} = \cos\theta,$$

$$\tan(-\theta) = \frac{P'M}{OM} = -\frac{y}{x} = -\frac{PM}{OM} = -\tan\theta$$

$\therefore$ sin (– 30°) = – sin 30°

cos (– 30°) = cos 30°

tan (– 30°) = – tan 30°.

Trigonometric Ratios of standard angles:-

Angles Trigonometrical ratios	**0°**	**30°**	**45°**	**60°**	**90°**
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cosec θ	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cot θ	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Trigonometrical ratios of :

90° – θ, 90° + θ, 180° + θ, 180° – θ, 360° – θ, 360° + θ.

We have,

sin (90° – θ) = cos θ $\quad$ cos (90° – θ) = sin θ

tan (90° – θ) = cot θ

sin (90° + θ) = cos θ $\quad$ cos (90° + θ) = – sin θ

tan (90° + θ) = – cot θ

sin (π – θ) = sin θ $\quad$ cos (π – θ) = – cos θ

tan (π – θ) = – tan θ

sin (π + θ) = – sin θ $\quad$ cos (π + θ) = – cos θ

tan (π + θ) = tan θ

sin (2π – θ) = – sin θ $\quad$ cos (2π + θ) = – cos θ

tan (2π – θ) = – tan θ

sin (2π + θ) = sin θ $\quad$ cos (2π + θ) = cos θ

tan (2π + θ) = tan θ

Example:

(1) State which of the following are positive ?

(i) cos 271° (ii) sec 75° (iii) cosec 159°

(iv) cos 315° (v) cot 375° (vi) tan 330°

(2) Find the value of A if cos 5A = sin 4A.

(3) Find the value of cos 24° + cos 5° + cos 175° + cos 204° + cos 300°.

(4) If cot A = tan (n – 1) A, find A.

(5) Find the domain of (i) tan θ and (ii) cot B.

(6) Answer the following questions:

(i) Find the value of

sin 480° cos 90° + cos 78° sin 1050°.

(ii) Express cot (θ – 1170°) as a function of θ.

(iii) Find the value of $\frac{\cot 54^\circ}{\tan 36^\circ}+\frac{\tan 20^\circ}{\cot 70^\circ}$.

(iv) Find the value of

$$\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\frac{5\pi}{16}+\cos^2\frac{7\pi}{16}.$$

(v) If sin x + sin² x + 1, then find the value of

$\cos^8 x + 2\cos^6 x + \cos^4 x$.

(vi) State the sign of sin 201° + cos 201°.

(vii) For what value(s) of p, the equation

$\sin\theta = x+\frac{p}{x}$ is meaningful for real x.

(viii) Find the maximum value of 6 sinx+8 cos x–3.

Example:

01. Evaluate :

(i) tan 1020° (ii) sec 480° (iii) tan (–945°)

(iv) cos 330° (v) sin 1230° (vi) cos (–1110°)

02. Express the following as the trigonometric ratio of an acute angle.

(i) sin 1000° (ii) tan 1140° (iii) cot (–1380°)

(iv) sec (– 890°) (v) cosec 730° (vi) cos (– 780°)

03. Find the value of

(i) sin 1° . sin 2° . sin 3° … sin 200°.

(ii) $\cot\frac{\pi}{20}\,.\,\cot\frac{3\pi}{20}\,.\,\cot\frac{5\pi}{20}\,.\,\cot\frac{7\pi}{20}\,.\,\cot\frac{9\pi}{20}$

04. Find the value of sin 480° cos 690° + cos 780° . sin 1050° + sin 780° sin 480° + cos 240° . cos 300°.

05. Find the value of (i) cot 1575° (ii) cos 1230°.

06. Show that

(i) $$\frac{\cos(180^\circ-A)\cot(90^\circ+A)\cos(-A)}{\tan(180^\circ+A)\tan(270^\circ+A)\sin(360^\circ-A)}=\cos A$$

(ii) $$\frac{\sin\left(\frac{\pi}{2}+\theta\right)\cos(\pi-\theta)\tan\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)\sin\left(\frac{3\pi}{2}-\theta\right)\tan\left(\frac{3\pi}{2}+\theta\right)}=1$$

(iii) $$\frac{\cot(\pi+\theta)\cot\left(\frac{\pi}{2}+\theta\right)\cos(4\pi-\theta)}{\tan\left(\frac{\pi}{2}-\theta\right)\operatorname{cosec}(\pi-\theta)\sin(-\theta)}$$

Formula:

(i) sin (A + B) = cos A sin B + sin A cos B

(ii) cos (A + B) = cos A cos B – sin A sin B

(iii) $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$.

(iv) sin (A–B) = sin A cos (– B) + cosA sin (– B)

= sin A . cos B – cos A . sin B

(v) cos (A–B) = cos A cos (–B) – sin A sin (–B)

= cos A . cos B + sin A . sin B

$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

(a) sin (A + B) sin (A – B) = sin²A – sin²B = cos² B – cos²A

(b) cos (A + B) cos (A – B) = cos²A – sin²B = cos²B – sin²A

(c) tan (A + B + C) =

$$\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

Example:

(i) Find the value of tan 75° , cot 105°.

(ii) Find the value of tan 75° . cot 105°.

(iii) If A + B + C = π, Prove that

tan A + tan B + tan C = tan A . tan B . tan C.

(iv) Prove that tan 75° + cot 75° = 4.

(v) Prove that $\frac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}=\tan 54^\circ$.

(vi) If $\tan A=\frac{1}{2}$ and $\tan B=\frac{1}{3}$, show that $A+B=\frac{\pi}{4}$ and cos 2A = sin 2B.

Formula:

(i) sin 2A = 2 sin A cos A

(ii) $\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$

1 + cos 2A = 2 cos²A, 1 – cos 2A = 2 sin²A,

$\frac{1-\cos 2A}{1+\cos 2A}=\tan^2 A$

(iii) tan 2A = tan (A + A)

(iv) sin 3A = 3 sin A – 4 sin³A

(v) cos 3A = 4 cos³ A – 3 cos A

(vi) $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$.

Formula:

(i) $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(iii) $1+\cos\theta = 2\cos^2\frac{\theta}{2}$

(iv) $1-\cos\theta = 2\sin^2\theta/2$

(v) $\dfrac{1-\cos\theta}{1+\cos\theta} = \tan^2\theta/2$

(vi) $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$

(vii) $\tan\theta = \dfrac{1-\cos\theta}{\sin\theta}$

(viii) $\cot\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{\sin\theta}$

(ix) $\sin\theta = 3\sin\theta/3 - 4\sin^3\theta/3.$

(x) $\cos\theta = 4\cos^3\theta/3 - 4\sin^3\theta/3.$

(xi) $\tan\theta = \dfrac{3\tan\frac{\theta}{3}-\tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}}.$

Transformation of sums and differences of trigonometrical ratios into their products and vice versa:

We know that

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

Adding and subtracting, we have

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ … (1)

$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$ … (2)

Similarly, we can have

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$ … (3)

$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$ … (4)

Put, $A+B = C$ and $A-B = D$

$\therefore A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$

Substituting in (1), (2), (3) and (4) we get

$\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ … (5)

$\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$ …(6)

$\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ …(7)

$\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2} = \sin\dfrac{C+D}{2}$

$= \sin\dfrac{D-C}{2}$ … (8)

LOGIC

Introduction

Logic is a field of study which deals with methods of reasoning. It provides rules by means of which we can determine whether a given argument or reasoning is valid or not. The Greek philosopher and Scientist Aristotle (381 – 322 BC) is said to be the first person to have studied logical reasoning. Logic is the Science of reasoning by which we arrive at a conclusion from known statements or assertions. It is also called science of accurate thinking.

Statement

A proposition or Statement is a declarative sentence which is either true or false but not both simultaneously. The truth or falsity of a statement is called its truth value. The truth values 'True' and 'False' of a statement are denoted by T and F respectively. They are also denoted by 1 and 0.

Connectives

Statements can be connected by words like 'and', 'or', etc. These words are known as logical connectives. The statements which do not contain any of the connectives are called simple statements. In other words, a simple statement cannot be broken down into two or more sentences.

Compound Statements

A statement that can be formed by combining two or more simple statements is called a compound statement.

Truth Table

The table showing the truth values of a statement formula is called "Truth Table".

Conjunction

A compound statement obtained by combining two simple statements say p and q, by using the connective "and" is called conjunction, i.e. the conjunction of two statements p and q is the statement $p \wedge q$. It is read as "p and q'.

Axiom of Conjunction :

The statement $P \wedge q$ has truth value T, whenever both p and q have truth value T, otherwise $p \wedge q$ has truth value F. The axiom of conjunction can be represented by following table.

Truth table of $p \wedge q$

P	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Any two statements can be combined by the connective "or" to form a statement, called the disjunction of the statements.

The disjunction of p and q is denoted symbolically by $p \vee q$. It is read as "p or q".

Axiom of Disjunction

The statement $p \vee q$ has truth value T if at least one of p and q has truth value T, otherwise $p \vee q$ has truth value F.

Truth table of $p \vee q$

P	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

The negation of a statement is formed by the use of the word 'not'. If p is any statement, then the denial of p is called negation of p. It is denoted by $\sim p$ (not p).

Axiom of negation :

For any statement p, if p is true then $\sim p$ is false and if p is false then $\sim p$ is true.

Note : A statement and its negation have opposite truth values.

Truth table of ~ P

P	~ P
T	F
F	T

Conditional Statements

If p and q are any two statements then the statement $p \rightarrow q$ is called a conditional statement. It is read as "if p then q". It can also be read as (i) p only if q (ii) p implies q (iii) p is sufficient for q (iv) q if p (v) q is necessary for p.

The statement p is called hypothesis (or antecedent) and the statement q is called conclusion (or consequent).

Axiom of conditional/implication :

A conditional $p \rightarrow q$ is false only when p is true and q is false, otherwise true.

Truth table of $p \rightarrow q$

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse, Inverse and Contrapositive

If $p \rightarrow q$ is a conditional statement, then

(a) $q \rightarrow p$ is called its converse

(b) $\sim p \rightarrow \sim q$ is called its inverse

(c) $\sim q \rightarrow \sim p$ is called its contrapositive

Truth table

p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Biconditional (Double implication)

A statement of the form "p if and only if q" is called a Biconditional statement. It is denoted by $p \leftrightarrow q$.

Axiom of Biconditional :

Biconditional $p \leftrightarrow q$ is True if p and q have same truth values, otherwise false.

Truth table of $p \leftrightarrow q$

p	q	$P \rightarrow q$	$q \rightarrow p$	$P \leftrightarrow q$
T	T	T	T	T
F	F	T	T	T
T	F	F	T	F
F	T	F	F	F

Equivalent statements

The statements p and q are equivalent if both have same truth value. It is denoted by $p \equiv q$ or $p \Leftrightarrow q$ or $p \leftrightarrow q$.

Note : $(p \rightarrow q) \wedge (q \rightarrow p) \equiv p \leftrightarrow q$

$P \rightarrow q \equiv \sim p \vee q$

$q \rightarrow p \equiv \sim q \vee p$

$\therefore p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Tautology

If a statement is always true irrespective of the truth values of its prime components, then it is called a tautology.

Contradiction (Fallacy)

A statement which is always false irrespective of the truth values of its prime components, is called contradiction.

Duality Principle

The principle of duality states that any established result involving statement and connectives $\vee$ and $\wedge$ gives a corresponding dual result if $\wedge$ is replaced by $\vee$ and $\vee$ is replaced by $\wedge$. Two statements P and P^* are said to be dual of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Eg. If p has truth value T, write the truth value of

(i) $\sim p \wedge q \rightarrow q \vee q$ (ii) $p \vee q \rightarrow \sim p \wedge q$

Eg. Determine the truth values of $p \leftrightarrow \sim q$ when the biconditional $p \leftrightarrow q$ has truth value :

(i) T (ii) F.

Eg. Prove by constructing truth table that

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $\sim (p \wedge q) \equiv \sim p \wedge \sim q$

(iii) $(p \rightarrow q) \equiv [\sim q \rightarrow (\sim p \wedge \sim q)]$

Answer the following question:

01. (i) Write down propositions p and q for which $q \rightarrow p$ is true but $p \leftrightarrow q$ is false.

(ii) Is the statement $[(p \rightarrow q) \wedge \sim q)] \wedge \sim p$ a tautology ?

(iii) Check whether the following sentence is a statement : "Every set is a finite set".

(iv) Give an example, with justification of a compound proposition that is neither a tautology nor a contradiction.

(v) Write the negation of $p \leftrightarrow q$.

(vi) Write the negation of the statement "for every real number x, $x^2 > x$" .

(vii) If $p \rightarrow q$ is false, then write the truth value of $\sim (p \wedge q) \rightarrow q$.

(viii) If $p \rightarrow q$ is true, then write the truth value of $\sim p \vee (p \rightarrow q)$.

(ix) If p is true then write the truth value of $(p \wedge q) \rightarrow p$.

(x) Write the truth value of $(p \vee q) \vee (\sim p \wedge \sim q)$.

02. Which of the following sentences are propositions and which are not ? State with reasons.

(i) There are 33 days in a month.

(ii) Answer this question.

(iii) Mathematics is difficult.

(iv) Today is Sunday.

(v) x is greater than 5.

(vi) Bring a glass of water.

(vii) Is 10 > 12 ?

03. Write the converse, inverse and contrapositive of each of the following propositions.

(i) $p \rightarrow \sim q$

(ii) If it rains, then he is happy.

(iii) If Δ ABC is right angled at B, then $AB^2+BC^2=AC^2$.

(iv) The square of an integer is a natural number.

(v) If the triangle is equilateral, then it is equiangular.

(vi) If x + 6 = 10 than x = 4

(vii) Sum of two odd integers is even.

04. If p is true and q is false, write the truth values of the following :

(i) $\sim (p \rightarrow \sim q)$

(ii) $(p \wedge q) \rightarrow (p \vee \sim (q \wedge \sim r))$

(iii) $[\sim (p \wedge q)] \vee [\sim (q \leftrightarrow p)]$

(iv) $(p \rightarrow q) \vee [\sim (p \rightarrow \sim q)]$

Eg: Show that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$,

$n \geq 1, n \in N$, by axiom of induction.

Eg. Prove that for every natural number $n \geq 1$, $10^n + 3.\ 4^{n+2} + 5$ is divisible by 9, by method of induction.

Prove the following by method of induction:

(i) $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}, n \geq 1$.

(ii) $1^3 + 2^3 + \ldots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2, n \geq 1$.

(iii) $1 + r + r^2 + \ldots + r^n = \dfrac{r^{n+1} - 1}{r-1}, r \neq 1$.

(iv) $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}, n \geq 1$.

(v) n (n + 1) (n + 5) is divisible by 6.

(vi) $4^n + 15n - 1$ is divisible by 9 for all natural numbers n.

(vii) $a^n - b^n$ is divisible by a – b, $n \in N$.

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